NOTE OF FUNCTIONAL ANALYSIS

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Throughout this note, all spaces $X, Y, ..$ are normed spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. And B_X denotes the closed unit ball of X.

1. Hahn-Banach Theorem

Lemma 1.1. *Let Y be a subspace of X and $v \in X \setminus Y$. Let $Z = Y \oplus \mathbb{K}v$ be the linear span of Y and v in X. If $f \in Y^*$, then there is an extension $F \in Z^*$ of f such that $\|F\| = \|f\|$.*

Proof. W.L.O.G. assume that $\|f\| = 1$.
Case $\mathbb{K} = \mathbb{R}$:
We first note that since $\|f\| = 1$, we have $|f(x) - f(y)| \leq \|(x + v) - (y + v)\|$ for all $x, y \in Y$. This implies that $-f(x) - \|x + v\| \leq -f(y) - \|y + v\|$ for all $x, y \in Y$. Now let $\gamma = \sup\{-f(x) - \|x + v\| : x \in X\}$. This implies that γ exists and

$$-f(y) - \|y + v\| \leq \gamma \leq -f(y) - \|y + v\| \tag{1.1}$$

for all $y \in Y$. So if we define $F : Z \longrightarrow \mathbb{R}$ by $F(y + \alpha v) := f(y) + \alpha\gamma$. It is clear that $F|_Y = f$. For showing $F \in Z^*$ with $\|F\| = 1$, since $F|_Y = f$ on Y and $\|f\| = 1$, it needs to show $|F(y + \alpha v)| \leq \|y + \alpha v\|$ for all $y \in Y$ and $\alpha \in \mathbb{R}$.
In fact, for $y \in Y$ and $\alpha > 0$, then by inequality 1.1, we have

$$|F(y + \alpha v)| = |f(y) + \alpha\gamma| \leq \|y + \alpha v\|. \tag{1.2}$$

Since y and α are arbitrary in inequality 1.2, we see that $|F(y + \alpha v)| \leq \|y + \alpha v\|$ for all $y \in Y$ and $\alpha \in \mathbb{R}$. Therefore the result holds when $\mathbb{K} = \mathbb{R}$.
Now for the complex case, let $h = \Re ef$ and $g = \Im mf$. Then $f = h + ig$ and f, g both are real linear with $\|h\| \leq 1$. Note that since $f(iy) = if(y)$ for all $y \in Y$. This implies that $g(y) = -ih(iy)$ for all $y \in Y$. This gives $f(\cdot) = h(\cdot) - ih(i\cdot)$ on Y. Then by the real case above, there is a real linear extension H on $Z := Y \oplus \mathbb{R}v \oplus i\mathbb{R}v$ of h such that $\|H\| = \|h\|$. Now define $F : Z \longrightarrow \mathbb{C}$ by $F(\cdot) := H(\cdot) - iH(i\cdot)$. Then $F \in Z^*$ and $F|_Y = f$. Thus it remains to show that $\|F\| = \|f\| = 1$. It needs to show that $|F(z)| \leq \|z\|$ for all $z \in Z$. Note for $z \in Z$, consider the polar form $F(z) = re^{i\theta}$. Then $F(e^{-i\theta}z) = r \in \mathbb{R}$ and thus $F(e^{-i\theta}z) = H(e^{-i\theta}z)$. This yields that

$$|F(z)| = r = |F(e^{-i\theta}z)| = |H(e^{-i\theta}z)| \leq \|H\|\|e^{-i\theta}z\| \leq \|z\|.$$

The proof is finished. □

Theorem 1.2. Hahn-Banach Theorem : *Let X be a normed space and let Y be a subspace of X. If $f \in Y^*$, then there exists a linear extension $F \in X^*$ of f such that $\|F\| = \|f\|$.*

Proof. Let $\mathfrak{X}$ be the collection of the pairs (Y, f), where Y is a subspace of X and $f \in Y^*$. Define a partial order $\leq$ on $\mathfrak{X}$ by $(Y_1, f_1) \leq (Y_2, f_2)$ if $Y_1 \subseteq Y_2$ and $f_2|_{Y_1} = f_1$. Then by the Zorn's lemma, there is a maximal element $(\widetilde{Y}, F)$ in $\mathfrak{X}$. Then by the maximality of $(\widetilde{Y}, F)$ and Lemma 1.1 will give $\widetilde{Y} = X$. The proof is finished. □

Proposition 1.3. *Let X be a normed space and $x_0 \in X$. Then there is $f \in X^*$ with $\|f\| = 1$ such that $f(x_0) = \|x_0\|$. Consequently, if $x, y \in X$ with $x \neq y$, then there exists $f(x) \neq f(y)$.*

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Proof. □

Proposition 1.4. *With the notation as above, if M is closed subspace and $v \in X \setminus M$, then there is $f \in X^*$ such that $f(M) \equiv 0$ and $f(v) \neq 0$.*

Proof. □

Proposition 1.5. *Retains the notation as above. If X^* is separable, then X is separable.*

Proof. Let $\{f_1, f_2....\}$ be a dense subset of X^*. Then there is a sequence (x_n) in X with $\|x_n\| = 1$ and $|f_n(x_n)| \geq 1/2\|f_n\|$ for all n. Now let M be the closed linear span of x_n's. Then M is a separable closed subspace of X. We are going to show that $M = X$.
Suppose not. Prop 1.4 will imply that there is a non-zero element $f \in X^*$ such that $f(M) \equiv 0$. On the other hand since $\{f_1, f_2....\}$ is dense in X^*, then there is a subsequence (f_{n_k}) such that $\|f_{n_k} - f\| \to 0$. This gives

$$\frac{1}{2}\|f_{n_k}\| \leq |f_{n_k}(x_{n_k})| = |f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \to 0$$

because $f(M) \equiv 0$. So $\|f_{n_k}\| \to 0$ and hence $f = 0$. It leads to a contradiction. □

Proposition 1.6. *Let X and Y be normed spaces. If $T \in \mathcal{L}(X, Y)$, then its adjoint operator $T^* \in \mathcal{L}(Y^*, X^*)$ and $\|T\| = \|T^*\|$.*

Proof. □

Proposition 1.7. *Let $Q : X \longrightarrow X^{**}$ be the canonical map. Then Q is an isometry.*

Proof. Note that for $x \in X$ and $x^* \in B_{X^*}$, we have $|Q(x)(x^*)| = |x^*(x)| \leq \|x\|$. Then $\|Q(x)\| \leq \|x\|$.
It remains to show that $\|x\| \leq \|Q(x)\|$ for all $x \in X$. In fact, for $x \in X$, there is $x^* \in X^*$ with $\|x^*\| = 1$ such that $\|x\| = |x^*(x)| = |Q(x)(x^*)|$ by Proposition 1.3. Thus we have $\|x\| \leq \|Q(x)\|$. The proof is finished. □

Definition 1.8. *A normed space X is said to be reflexive if the canonical map $Q : X \longrightarrow X^{**}$ is surjective.*

Example 1.9. *(i) : ℓ^p is reflexive for $1 < p < \infty$.*
(ii) : c_0 is not reflexive.

Proposition 1.10. *Every subspace (not necessary to be closed) of a reflexive space is reflexive.*

Proof. □

2. Open Mapping Theorem

Lemma 2.1. *Let $T : X \longrightarrow Y$ be a bounded linear surjection from a Banach space X onto a Banach space Y. Then 0 is an interior point of $T(U(1))$, where $U(r) := \{x \in X : \|x\| < r\}$ for $r > 0$.*

Proof. **Claim 1** : 0 is an interior point of $\overline{T(U(1))}$.
Note that since T is surjective, $Y = \bigcup_{n=1}^{\infty} T(U(n))$. Then by the second category theorem, there exists N such that *int* $\overline{T(U(N))} \neq \emptyset$. Let y' be an interior point of $\overline{T(U(N))}$. Then there is $\eta > 0$ such that $B(y', \eta) \subseteq \overline{T(U(N))}$. Since $B(y', \eta) \cap T(U(N)) \neq \emptyset$, we may assume that $y' \in T(U(N))$. Let $x' \in U(N)$ such that $T(x') = y'$. Then we have

$$0 \in B(y', \eta) - y' \subseteq \overline{T(U(N))} - T(x') \subseteq \overline{T(U(2N))} = 2N\overline{T(U(1))}.$$

So we have $0 \in \frac{1}{2N}(B(y', \eta) - y') \subseteq \overline{T(U(1))}$. Hence 0 is an interior point of $\overline{T(U(1))}$. So Claim 1 follows.
Therefore there is $r > 0$ such that $B_Y(0, r) \subseteq \overline{T(U(1))}$. This implies that we have

$$B_Y(0, r/2^k) \subseteq \overline{T(U(1/2^k))} \tag{2.1}$$

for all $k = 0, 1, 2....$
Claim 2 : $D := B_Y(0, r) \subseteq T(U(3))$.
Let $y \in D$. By Eq 2.1, there is $x_1 \in U(1)$ such that $\|y - T(x_1)\| < r/2$. Then by using Eq 2.1 again, there is $x_2 \in U(1/2)$ such that $\|y - T(x_1) - T(x_2)\| < r/2^2$. To repeat the same steps, there exists is a sequence (x_k) such that $x_k \in U(1/2^{k-1})$ and

$$\|y - T(x_1) - T(x_2) - ... - T(x_k)\| < r/2^k$$

for all k. On the other hand, since $\sum \|x_k\| < \sum 1/2^k$ and X is Banach, $x := \sum_{k=1}^{\infty} x_k$ exists in X and $\|x\| \leq \sum_{k=1}^{\infty} 1/2^{k-1} = 2$. This implies that $y = T(x)$ and $\|x\| < 3$.
Thus we the result follows. □

Theorem 2.2. Open Mapping Theorem : *Retains the notation as in Lemma 2.1. Then T is an open mapping.*

Proof. By Lemma 2.1, we first note that for every $r > 0$, there is $\delta > 0$ such that $B_Y(0, \delta) \subseteq T(B_X(0, r))$. Let U be an open subset of X and let $b \in T(U)$. We need to show that b is an interior point of $T(U)$. Let $a \in U$ such that $T(a) = b$. Since U is open, there is $r > 0$ such that $a + B(0, r) = B(a, r) \subseteq U$. Then $B(0, r) = B(a, r) - a$. Choosing $\delta > 0$ as above, hence we have

$$B(0, \delta) \subseteq T(B(0, r)) = T(B(a, r)) - T(a) \subseteq T(U) - b.$$

This implies that $b + B(0, r) \subseteq T(U)$. The proof is finished. □

Proposition 2.3. *Let T be a bounded linear isomorphism between Banach spaces X and Y. Then T^{-1} must be bounded. Consequently, there are $c_1, c_2 > 0$ such that $c_1\|\cdot\| \leq \|T(\cdot)\| \leq c_2\|\cdot\|$ on X.*

3. Closed Graph Theorem

Let $T : X \longrightarrow Y$. The *graph* of T, write $\mathcal{G}(T)$ is defined by the set $\{(x, y) \in X \times Y : y = T(x)\}$. Now the direct sum $X \oplus Y$ is endowed with the norm $\|\cdot\|_\infty$, that is $\|x \oplus y\|_\infty := \max(\|x\|_X, \|y\|_Y)$. We also write $X \oplus_\infty Y$ when $X \oplus Y$ is equipped with this norm.
We say that an operator $T : X \longrightarrow Y$ is said to be closed if its graph $\mathcal{G}(T)$ is a closed subset of $X \oplus_\infty Y$, that is, if a sequence (x_n) of X satisfying the condition $\|(x_n, Tx_n) - (x, y)\|_\infty \to 0$ for some $x \in X$ and $y \in Y$ implies $T(x) = y$.

Theorem 3.1. Closed Graph Theorem : *Let $T : X \longrightarrow Y$ be a linear operator from a Banach space X to a Banach Y. Then T is bounded if and only if T is closed.*

Proof. The part $(\Rightarrow)$ is clear.
Assume that T is closed, that is, the graph $\mathcal{G}(T)$ is $\|\cdot\|_\infty$-closed. Define $\|\cdot\|_0 : X \longrightarrow [0, \infty)$ by

$$\|x\|_0 = \|x\| + \|T(x)\|$$

for $x \in X$. Then $\|\cdot\|_0$ is a norm on X. Let $I : (X, \|\cdot\|_0) \longrightarrow (X, \|\cdot\|)$ be the identity operator. It is clear that I is bounded since $\|\cdot\| \leq \|\cdot\|_0$.
Claim: $(X, \|\cdot\|_0)$ is Banach. In fact, let (x_n) be a Cauchy sequence in $(X, \|\cdot\|_0)$. Then (x_n) and $(T(x_n))$ both are Cauchy sequences in $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$. Since X and Y are Banach spaces, there are $x \in X$ and $y \in Y$ such that $\|x_n - x\|_X \to 0$ and $\|T(x_n) - y\|_Y \to 0$. Thus $y = T(x)$ since the graph $\mathcal{G}(T)$ is closed.
Therefore $I^{-1} : (X, \|\cdot\|) \longrightarrow (X, \|\cdot\|_0)$ is bounded by Proposition 2.3. Thus if $x_n \to 0$ in $(X, \|\cdot\|)$, then $T(x_n) \to 0$ in Y. So, T is bounded. The proof is finished. □

Example 3.2. *Let $D := \{\mathbf{c} = (c_n) \in \ell^2 : \sum n^2|c_n|^2 < \infty\}$. Define $T : D \longrightarrow \ell^2$ by $T(\mathbf{c}) = (nc_n)$. Then T is an unbounded closed operator.*

Proof. Note that since $\|Te_n\| = n$ for all n, T is not bounded. Now we claim that T is closed.
Let $(\mathbf{x_i})$ be a convergent sequence in D such that $(T\mathbf{x_i})$ is also convergent in ℓ^2. Write $\mathbf{x_i} = (x_{i,n})_{n=1}^{\infty}$ with $\lim_i \mathbf{x_i} = \mathbf{x} := (x_n)$ in D and $\lim_i T\mathbf{x_i} = \mathbf{y} := (y_n)$ in ℓ^2. This implies that if we fix n_0, then

$\lim_i x_{i,n_0} = x_{n_0}$ and $\lim_i n_0 x_{i,n_0} = y_{n_0}$. This gives $n_0 x_{n_0} = y_{n_0}$. Thus $T\mathbf{x} = \mathbf{y}$ and hence T is closed. $\square$

Example 3.3. *Let $X := \{f \in C^b(0,1) \cap C^\infty(0,1) : f' \in C^b(0,1)\}$. And X is endowed with $\|\cdot\|_\infty$. Define $T : f \in X \mapsto f' \in C^b(0,1)$. Then T is a closed unbounded operator.*

Proof. Note that if a sequence $f_n \to f$ in X and $f_n' \to g$ in $C^b(0,1)$. Then $f' = g$. Hence T is closed. In fact, if we fix some $0 < c < 1$, then by the Fundamental Theorem of Calculus, we have

$$0 = \lim_n (f_n(x) - f(x)) = \lim_n (\int_c^x (f_n'(t) - f'(t))dt) = \int_c^x (g(t) - f'(t))dt$$

for all $x \in (0,1)$. This implies that we have $\int_c^x g(t)dt = \int_c^x f'(t)dt$. So $g = f'$ on $(0,1)$.
On the other hand, since $\|Tx^n\|_\infty = n$ for all $n \in \mathbb{N}$. Thus T is unbounded as desired. $\square$

4. Uniform Boundedness Theorem

Theorem 4.1. Uniform Boundedness Theorem : *Let $\{T_i : X \longrightarrow Y : i \in I\}$ be a family of bounded linear operators from a Banach space X into a normed space Y. Suppose that for each $x \in X$, we have $\sup_{i\in I} \|T_i(x)\| < \infty$. Then $\sup_{i\in I} \|T_i\| < \infty$.*

Proof. For each $x \in X$, define

$$\|x\|_0 := \max(\|x\|, \sup_{i\in I} \|T_i(x)\|).$$

Then $\|\cdot\|_0$ is a norm on X and $\|\cdot\| \le \|\cdot\|_0$ on X. If $(X, \|\cdot\|_0)$ is complete, then by the Open Mapping Theorem. This implies that $\|\cdot\|$ is equivalent to $\|\cdot\|_0$ and thus there is $c > 0$ such that

$$\|T_j(x)\| \le \sup_{i\in I} \|T_i(x)\| \le \|x\|_0 \le c\|x\|$$

for all $x \in X$ and for all $j \in I$. So $\|T_j\| \le c$ for all $j \in I$ is as desired.
Thus it remains to show that $(X, \|\cdot\|_0)$ is complete. In fact, if (x_n) is a Cauchy sequence in $(X, \|\cdot\|_0)$, then it is also a Cauchy sequence with respect to the norm $\|\cdot\|$ on X. Write $x := \lim_n x_n$ with respect to the norm $\|\cdot\|$. Also for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|T_i(x_n - x_m)\| < \varepsilon$ for all $m, n \ge N$ and for all $i \in I$. Now fixing $i \in I$ and $n \ge N$ and taking $m \to \infty$, we have $\|T_i(x_n - x)\| \le \varepsilon$ and thus $\sup_{i\in I} \|T_i(x_n - x)\| \le \varepsilon$ for all $n \ge N$. So we have $\|x_n - x\|_0 \to 0$ and hence $(X, \|\cdot\|_0)$ is complete. The proof is finished. $\square$

Remark 4.2. *Consider $c_{00} := \{\mathbf{x} = (x_n) : \exists\ N, \forall\ n \ge N; x_n \equiv 0\}$ which is endowed with $\|\cdot\|_\infty$. Now for each $k \in \mathbb{N}$, if we define $T_k \in c_{00}^*$ by $T_k((x_n)) := kx_k$, then $\sup_k |T_k(\mathbf{x})| < \infty$ for each $\mathbf{x} \in c_{00}$ but $(\|T_k\|)$ is not bounded, in fact, $\|T_k\| = k$. Thus the assumption of the completeness of X in Theorem 4.1 is essential.*

Corollary 4.3. *Let X and Y be as in Theorem 4.1. Let $T_k : X \longrightarrow Y$ be a sequence of bounded operators. Assume that $\lim_k T_k(x)$ exists in Y for all $x \in X$. Then there is $T \in \mathcal{L}(X,Y)$ such that $\lim_k \|(T - T_k)x\| = 0$ for all $x \in X$. Moreover, we have $\|T\| \le \liminf_k \|T_k\|$.*

5. Weakly convergent and Weak* convergent

Proposition 5.1. *Let (x_n) be a sequence in a normed space X. If $\lim_n f(x_n)$ exists for all $f \in X^*$, then (x_n) is a bounded sequence.*

Remark 5.2. *Note that although $\lim_n f(x_n)$ exists for all $f \in X^*$ in Corollary 5.2, it does not imply that (x_n) is convergent.*
For example, consider $X = c_0$ and (e_n). Then $f(e_n) \to 0$ for all $f \in c_0^* = \ell^1$ but (e_n) is not convergent in c_0.

Definition 5.3. *Let X be a normed space. A sequence (x_n) is said to be weakly convergent if there is $x \in X$ such that $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, x is called a weak limit of (x_n).*

Proposition 5.4. *A weak limit is unique. In this case, if (x_n) weakly converges to x, write $x = w\text{-}\lim_n x_n$ or $x_n \xrightarrow{w} x$.*

Remark 5.5. *It is clear that if a sequence (x_n) converges to $x \in X$ in norm, then $x_n \xrightarrow{w} x$. However, the converse does not hold following from Remark 5.2.*

Proposition 5.6. *Suppose that X is finite dimensional. A sequence (x_n) in X is norm convergent if and only if it is weakly convergent.*

Proof. Suppose that (x_n) weakly converges to x. Let $\mathcal{B} := \{e_1, .., e_N\}$ be a base for X and let f_k be the k-th coordinate functional corresponding to the base $\mathcal{B}$, that is $v = \sum_{k=1}^{N} f_k(v)e_k$ for all $v \in X$. Since $\dim X < \infty$, we have f_k in X^* for all $k = 1, ..., N$. Therefore, we have $\lim_n f_k(x_n) = f_k(x)$ for all $k = 1, ..., N$. So, we have $\|x_n - x\| \to 0$. □

Definition 5.7. *Let X be a normed space. A sequence (f_n) in X^* is said to be weak* convergent if there is $f \in X^*$ such that $\lim_n f_n(x) = f(x)$ for all $x \in X$, that is f_n point-wise converges to f. In this case, f is called the weak* limit of (f_n). Write $f = w^*\text{-}\lim_n f_n$ or $f_n \xrightarrow{w^*} f$.*

Remark 5.8. *In the dual space X^* of a normed space X, we always have the following implications:*

$$\text{“Norm Convergent”} \Longrightarrow \text{“Weakly Convergent”} \Longrightarrow \text{“Weak}^* \text{ Convergent”}.$$

However, the converse of each implication does not hold.

Example 5.9. *Remark 5.2 has shown that the w-convergence does not imply $\|\cdot\|$-convergence.*
We now claim that the w^*-convergence Does Not imply the w-convergence.
Consider $X = c_0$. Then $c_0^* = \ell^1$ and $c_0^{**} = (\ell^1)^* = \ell^\infty$. Let $e_n^* = (0, ...0, 1, 0...) \in \ell^1 = c_0^*$, where the n-th coordinate is 1. Then $e_n^* \xrightarrow{w^*} 0$ but $e_n^* \nrightarrow 0$ weakly because $e^{**}(e_n^*) \equiv 1$ for all n, where $e^{**} := (1, 1, ...) \in \ell^\infty = c_0^{**}$. Hence the w^*-convergence does not imply the w-convergence.

Proposition 5.10. *Let (f_n) be a sequence in X^*. Suppose that X is reflexive. Then $f_n \xrightarrow{w} f$ if and only if $f_n \xrightarrow{w^*} f$.*
In particular, if $\dim X < \infty$, then the followings are equivalent:

(i) : $f_n \xrightarrow{\|\cdot\|} f_n$;
(ii) : $f_n \xrightarrow{w} f_n$;
(iii) : $f_n \xrightarrow{w^*} f_n$.

Proposition 5.11. (**Banach**) : *Let X be a separable normed space. If (f_n) is a bounded sequence in X^*, then it has a w^*-convergent subsequence.*

Proof. Let $D := \{x_1, x_2, ...\}$ be a countable dense subset of X. Note that since $(f_n)_{n=1}^\infty$ is bounded, $(f_n(x_1))$ is a bounded sequence in $\mathbb{K}$. Then $(f_n(x_1))$ has a convergent subsequence, say $(f_{1,k}(x_1))_{k=1}^\infty$ in $\mathbb{K}$. Let $f(x_1) := \lim_k f_{1,k}(x_1)$. Now consider the bounded sequence $(f_{1,k}(x_2))$. Then there is convergent subsequence, say $(f_{2,k}(x_2))$, of $(f_{1,k}(x_2))$. Put $f(x_2) := \lim_k f_{2,k}(x_2)$. Notice that we still have $f(x_1) = \lim_k f_{2,k}(x_1)$. To repeat the same step, if we define $(m, k) \le (m', k')$ if $m < m'$; or $m = m'$ with $k \le k'$, we can find a sequence $(f_{m,k})_{m,k}$ in X^* such that

(i) : $(f_{m+1,k})_{k=1}^\infty$ is a subsequence of $(f_{m,k})_{k=1}^\infty$ for $m = 0, 1, ..,$ where $f_{0,k} := f_k$.
(ii) : $f(x_i) = \lim_k f_{m,k}(x_i)$ exists for all $1 \le i \le m$.

Now put $h_m := f_{m,m}$. Then (h_m) is a subsequence of (f_n). Notice that for each i, we have $\lim_m h_m(x_i) = \lim_m f_{i,m}(x_i) = f(x_i)$ by the construction (ii) above. Since $(\|h_m\|)$ is bounded and D is dense in X, we have $h(x) := \lim_m h_m(x)$ exists for all $x \in X$ and $h \in X^*$. That is $h = w^*\text{-}\lim_m h_m$. The proof is finished. □

Remark 5.12. *Theorem 5.11 does not hold if the separability of X is removed. For example, consider $X = \ell^\infty$ and δ_n the n-th coordinate functional on ℓ^∞. Then $\delta_n \in (\ell^\infty)^*$ with $\|\delta_n\| = 1$ for all n. Suppose that (δ_n) has a w^*-convergent subsequence $(\delta_{n_k})_{k=1}^\infty$. Define $\mathbf{x} = (x_m) \in \ell^\infty$ by*

$$x_m = \begin{cases} 0 & \text{if } m \neq n_k; \\ 1 & \text{if } m = n_{2k}; \\ -1 & \text{if } m = n_{2k+1}. \end{cases}$$

Hence we have $|\delta_{n_i}(\mathbf{x}) - \delta_{n_{i+1}}(\mathbf{x})| = 2$ for all $i = 1, 2, ...$ It leads to a contradiction. So (δ_n) has no w^*-convergent subsequence.

Corollary 5.13. *Let X be a separable space. In X^* assume that the set of all w^*-convergent sequences coincides with the set of all normed convergent sequences, that is a sequence (f_n) is w^*-convergent if and only if it is norm convergent. Then $\dim X < \infty$.*

Proof. It needs to show that the closed unit ball B_{X^*} in X^* is compact in norm. Let (f_n) be a sequence in B_{X^*}. Using Theorem 5.11, (f_n) has a w^*-convergent subsequence (f_{n_k}). Then by the assumption, (f_{n_k}) is norm convergent. Note that if $\lim_k f_{n_k} = f$ in norm, then $f \in B_{X^*}$. So B_{X^*} is compact and thus $\dim X^* < \infty$. So $\dim X^{**} < \infty$ that gives $\dim X$ is finite because $X \subseteq X^{**}$. □

6. Geometry of Hilbert space I

We first recall the following useful properties of an inner product space:

Proposition 6.1. *Let V be an inner product space. For all $x, y \in V$, we always have:*

- *(i):* (**Cauchy-Schwarz inequality**)*: $|(x,y)| \leq \|x\|\|y\|$ Consequently, the inner product on $V \times V$ is jointly continuous.*
- *(ii):* (**Parallelogram law**)*: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$*

Furthermore, a norm $\|\cdot\|$ on a vector space is induced by an inner product if and only if it satisfies the Parallelogram law.

Proposition 6.2. (**Bessel's inequality**) : *Let $\{e_1, ..., e_N\}$ be an orthonormal set in an inner product space V. Then for any $x \in V$, we have*

$$\sum_{i=1}^N |(x, e_i)|^2 \leq \|x\|^2.$$

Proof. It can be obtained by the following equality immediately

$$\|x - \sum_{i=1}^N (x, e_i) e_i\|^2 = \|x\|^2 - \sum_{i=1}^N |(x, e_i)|^2.$$

□

Corollary 6.3. *Let $(e_i)_{i \in I}$ be an orthonormal set in an inner product space V. Then for any element $x \in V$, the set*

$$\{i \in I : (e_i, x) \neq 0\}$$

is countable.

Proof. Note that for each $x \in V$, we have

$$\{i \in I : (e_i, x) \neq 0\} = \bigcup_{n=1}^\infty \{i \in I : |(e_i, x)| \geq 1/n\}.$$

Then the Bessel's inequality implies that the set $\{i \in I : |(e_i, x)| \geq 1/n\}$ must be finite for each $n \geq 1$. So the result follows. □

In the rest of this section, X always denotes a complex Hilbert space with an inner product $(\cdot, \cdot)$.

Proposition 6.4. *Let (e_n) be a sequence of orthonormal vectors in a Hilbert space X. Then for any $x \in V$, the series $\sum_{n=1}^{\infty}(x, e_n)e_n$ is convergent.*
Moreover, if $(e_{\sigma(n)})$ is a rearrangement of (e_n), that is, $\sigma : \{1, 2...\} \longrightarrow \{1, 2, ..\}$ is a bijection. Then we have

$$\sum_{n=1}^{\infty}(x, e_n)e_n = \sum_{n=1}^{\infty}(x, e_{\sigma(n)})e_{\sigma(n)}.$$

Proof. Since X is a Hilbert space, the convergence of the series $\sum_{n=1}^{\infty}(x, e_n)e_n$ follows from the Bessel's inequality at once. In fact, if we put $s_p := \sum_{n=1}^{p}(x, e_n)e_n$, then we have

$$\|s_{p+k} - s_p\|^2 = \sum_{p+1 \leq n \leq p+k} |(x, e_n)|^2.$$

Now put $y = \sum_{n=1}^{\infty}(x, e_n)e_n$ and $z = \sum_{n=1}^{\infty}(x, e_{\sigma(n)})e_{\sigma(n)}$. Notice that we have

$$\begin{aligned}(y, y - z) &= \lim_N (\sum_{n=1}^{N}(x, e_n)e_n, \sum_{n=1}^{N}(x, e_n)e_n - z) \\ &= \lim_N \sum_{n=1}^{N} |(x, e_n)|^2 - \lim_N \sum_{n=1}^{N}(x, e_n) \sum_{j=1}^{\infty} \overline{(x, e_{\sigma(j)})}(e_n, e_{\sigma(j)}) \\ &= \sum_{n=1}^{\infty} |(x, e_n)|^2 - \lim_N \sum_{n=1}^{N}(x, e_n)\overline{(x, e_n)} \quad \text{(N.B: for each } n\text{, there is a unique } j \text{ such that } n = \sigma(j)) \\ &= 0.\end{aligned}$$

Similarly, we have $(z, y - z) = 0$. The result follows. □

A family of an orthonormal vectors, say $\mathcal{B}$, in X is said to be **complete** if it is maximal with respect to the set inclusion order,that is, if $\mathcal{C}$ is another family of orthonormal vectors with $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$.
A complete orthonormal subset of X is also called an **orthonormal base** of X.

Proposition 6.5. *Let $\{e_i\}_{i \in I}$ be a family of orthonormal vectors in X. Then the followings are equivalent:*

(i): $\{e_i\}_{i \in I}$ is complete;
(ii): if $(x, e_i) = 0$ for all $i \in I$, then $x = 0$;
(iii): for any $x \in X$, we have $x = \sum_{i \in I}(x, e_i)e_i$;
(iv): for any $x \in X$, we have $\|x\|^2 = \sum_{i \in I} |(x, e_i)|^2$.

In this case, the expression of each element $x \in X$ in Part (iii) is unique.

Note : *there are only countable many $(x, e_i) \neq 0$ by Corollary 6.3, so the sums in (iii) and (iv) are convergent by Proposition 6.4.*

Proposition 6.6. *Let X be a Hilbert space. Then*

(i) : X processes an orthonormal base.
(ii) : If $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ both are the orthonormal bases for X, then I and J have the same cardinality. In this case, the cardinality $|I|$ of I is called the orthonormal dimension of X.

Proof. Part (i) follows from Zorn's Lemma at once.
For part (ii), if the cardinality $|I|$ is finite, then the assertion is clear since $|I| = \dim X$ (*vector space dimension*) in this case.

Now assume that $|I|$ is infinite, for each e_i, put $J_{e_i} := \{j \in J : (e_i, f_j) \neq 0\}$. Note that since $\{e_i\}_{i\in I}$ is maximal, Proposition 6.5 implies that we have

$$\{f_j\}_{j\in J} \subseteq \bigcup_{i\in I} J_{e_i}.$$

On the other hand, we have seen that We have seen that J_{e_i} is countable for each e_i. So we have $|\mathbb{N}| \leq |I|$ because $|I|$ is infinite and thus $|\mathbb{N} \times I| = |I|$. Then we have

$$|J| \leq \sum_{i\in I} |J_{e_i}| = \sum_{i\in I} |\mathbb{N}| = |\mathbb{N} \times I| = |I|.$$

From symmetry argument, we also have $|I| \leq |J|$. □

Remark 6.7. *Recall that a vector space dimension of X is defined by the cardinality of a maximal linearly independent set in X.*
Notice that if X is finite dimensional, then the orthonormal dimension is the same as the vector space dimension.
Also, the vector space dimension is larger than the orthornormal dimension in general since every orthogonal set must be linearly independent.

We say that two Hilbert spaces X and Y are said to be isomorphic if there is linear isomorphism U from X onto Y such that $(Ux, Ux') = (x, x')$ for all $x, x' \in X$. In this case U is called a unitary operator.

Theorem 6.8. *Two Hilbert spaces are isomorphic if and only if they have the same orthonornmal dimension.*

Proof. The converse part ($\Leftarrow$) is clear.
Now for the ($\Rightarrow$) part, let X and Y be isomorphic Hilbert spaces. Let $U : X \longrightarrow Y$ be a unitary. Note that if $\{e_i\}_{i\in I}$ is an orthonormal base of X, then $\{Ue_i\}_{i\in I}$ is also an orthonormal base of Y. Thus the necessary part follows from Proposition 6.6 at once. □

Corollary 6.9. *Every separable Hilbert space is isomorphic to ℓ^2 or $\mathbb{C}^n$ for some n.*

Proof. Let X be a separable Hilbert space.
If $\dim X < \infty$, then it is clear that X is isomorphic to $\mathbb{C}^n$ for $n = \dim X$.
Now suppose that $\dim X = \infty$ and its orthonormal dimension is larger than $|\mathbb{N}|$, that is X has an orthonormal base $\{f_i\}_{i\in I}$ with $|I| > |\mathbb{N}|$. Note that since $\|f_i - f_j\| = \sqrt{2}$ for all $i, j \in I$ with $i \neq j$. This implies that $B(e_i, 1/4) \cap B(e_j, 1/4) = \emptyset$ for $i \neq j$.
On the other hand, if we let D be a countable dense subset of X, then $B(f_i, 1/4) \cap D \neq \emptyset$ for all $i \in I$. So for each $i \in I$, we can pick up an element $x_i \in D \cap B(f_i, 1/4)$. Therefore, one can define an injection from I into D. It is absurd to the countability of D. □

7. Geometry of Hilbert space II

In this section, let X always denote a complex Hilbert space.

Proposition 7.1. *If D is a closed convex subset of X, then there is a unique element $z \in D$ such that*

$$\|z\| = \inf\{\|x\| : x \in D\}.$$

Consequently, for any element $u \in X$, there is a unique element $w \in D$ such that

$$\|u - w\| = d(u, D) := \inf\{\|u - x\| : x \in D\}.$$

Proof. We first claim the existence of such z.
Let $d := \inf\{\|x\| : x \in D\}$. Then there is a sequence (x_n) in D such that $\|x_n\| \to d$. Notice that (x_n) is a Cauchy sequence. In fact, the Parallelogram Law implies that

$$\|\frac{x_m - x_n}{2}\|^2 = \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - \|\frac{x_m + x_n}{2}\|^2 \leq \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - d \longrightarrow 0$$

as $m, n \to \infty$, where the last inequality holds because D is convex and hence $\frac{1}{2}(x_m + x_n) \in D$. Let $z := \lim_n x_n$. Then $\|z\| = d$ and $z \in D$ because D is closed.
For the uniqueness, let $z, z' \in D$ such that $\|z\| = \|z'\| = d$. Thanks to the Parallelogram Law again, we have

$$\|\frac{z - z'}{2}\|^2 = \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - \|\frac{z + z'}{2}\|^2 \leq \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - d = 0.$$

Therefore $z = z'$.
The last assertion follows by considering the closed convex set $u - D$ immediately. □

Proposition 7.2. *Suppose that M is a closed subspace. Let $u \in X$ and $w \in M$. Then the followings are equivalent:*

(i): $\|u - w\| = d(u, M)$;
(ii): $u - w \perp M$, that is $(u - w, x) = 0$ for all $x \in M$.

Consequently, for each element $u \in X$, there is a unique element $w \in M$ such that $u - w \perp M$.

Proof. Let $d := d(u, M)$.
For proving $(i) \Rightarrow (ii)$, fix an element $x \in M$. Then for any $t > 0$, note that since $w + tx \in M$, we have

$$d^2 \leq \|u - w - tx\|^2 = \|u - w\|^2 + \|tx\|^2 - 2Re(u - w, tx) = d^2 + \|tx\|^2 - 2Re(u - w, tx).$$

This implies that

$$2Re(u - w, x) \leq t\|x\|^2 \tag{7.1}$$

for all $t > 0$ and for all $x \in M$. So by considering $-x$ in Eq.7.1, we obtain

$$2|Re(u - w, x)| \leq t\|x\|^2.$$

for all $t > 0$. This implies that $Re(u - w, x) = 0$ for all $x \in M$. Similarly, putting $\pm ix$ into Eq.7.1, we have $Im(u - w, x) = 0$. So (ii) follows.
For $(ii) \Rightarrow (i)$, we need to show that $\|u - w\|^2 \leq \|u - x\|^2$ for all $x \in M$. Note that since $u - w \perp M$ and $w \in M$, we have $u - w \perp w - x$ for all $x \in M$. This gives

$$\|u - x\|^2 = \|(u - w) + (w - x)\|^2 = \|u - w\|^2 + \|w - x\|^2 \geq \|u - w\|^2.$$

Part (i) follows.
The last statement is obtained by Proposition 7.1 immediately. □

Theorem 7.3. *Let M be a closed subspace. Put*

$$M^\perp := \{x \in X : x \perp M\}.$$

Then $M^\perp$ is a closed subspace and we have $X = M \oplus M^\perp$.
In this case, $M^\perp$ is called the orthogonal complement of M.

Proof. It is clear that $M^\perp$ is a closed subspace and $M \cap M^\perp = (0)$. It remains to show $X = M + M^\perp$. Let $u \in X$. Then by Proposition 7.2, we can find an element $w \in M$ such that $u - w \perp M$. Thus $u - w \in M^\perp$ and $u = w + (u - w)$. The proof is finished. □

Corollary 7.4. *With the notation as above, an element $x_0 \notin M$ if and only if there is an element $m \in M$ such that $x_0 - m \perp M$.*

Proof. It is clear from Theorem 7.3. □

Corollary 7.5. *If M is a closed subspace of X, then $M^{\perp\perp} = M$.*

Proof. It is clear that $M \subseteq M^{\perp\perp}$ by the definition of $M^{\perp\perp}$. Now if there is $x \in M^{\perp\perp} \setminus M$, then by the decomposition $X = M \oplus M^{\perp}$ obtained in Theorem7.3, we have $x = y + z$ for some $y \in M$ and $z \in M^{\perp}$. This implies that $z = x - y \in M^{\perp} \cap M^{\perp\perp} = (0)$. This gives $x = y \in M$. It leads to a contradiction. □

Remark 7.6. *It is worthwhile pointing out that for a general Banach space X and a closed subspace M of X, it* **May Not** *have a complementary* **Closed** *subspace N of M, that is $X = M \oplus N$. If M has a complementary closed subspace X, we say that M is complemented in X.*

Example 7.7. (**Very Not Obvious** !!!) *c_0 is not complemented in ℓ^{∞}.*

8. Riesz Representation Theorem

Let X be a complex Hilbert space as before.

Theorem 8.1. Riesz Representation Theorem : *For each $f \in X^*$, then there is a unique element $v_f \in X$ such that*

$$f(x) = (x, v_f)$$

for all $x \in X$ and we have $\|f\| = \|v_f\|$.
Furthermore, if $(e_i)_{i\in I}$ is an orthonormal base of X, then $v_f = \sum_i \overline{f(e_i)} e_i$.

Proof. We first prove the uniqueness of v_f. If $z \in X$ also satisfies the condition: $f(x) = (x, z)$ for all $x \in X$. This implies that $(x, z - v_f) = 0$ for all $x \in X$. So $z - v_f = 0$.
Now for proving the existence of v_f, it suffices to show the case $\|f\| = 1$. Then $\ker f$ is a closed proper subspace. Then by the orthogonal decomposition again, we have

$$X = \ker f \oplus (\ker f)^{\perp}.$$

Since $f \neq 0$, we have $(\ker f)^{\perp}$ is linear isomorphic to $\mathbb{C}$. Also note that the restriction of f on $(\ker f)^{\perp}$ is of norm one. Hence there is an element $v_f \in (\ker f)^{\perp}$ with $\|v_f\| = 1$ such that $f(v_f) = \|f|_{(\ker f)^{\perp}}\| = 1$ and $(\ker f)^{\perp} = \mathbb{C} v_f$. So for each element $x \in X$, we have $x = z + \alpha v_f$ for some $z \in \ker f$ and $\alpha \in \mathbb{C}$. Then $f(x) = \alpha f(v_f) = \alpha = (x, v_f)$ for all $x \in X$.
Concerning about the last assertion, if we put $v_f = \sum\limits_{i\in I} \alpha_i e_i$, then $f(e_j) = (e_j, v_f) = \overline{\alpha_j}$ for all $j \in I$. The proof is finished. □

Corollary 8.2. *With the notation as in Theorem 8.1, Define the map*

$$\Phi : f \in X^* \mapsto v_f \in X,$$

that is $f(x) = (x, \Phi(f))$ for all $x \in X$ and $f \in X^*$.
And if we define $(f, g)_{X^*} := (v_g, v_f)_X$ for $f, g \in X^*$. Then $(X^*, (\cdot,\cdot)_{X^*})$ becomes a Hilbert space. Moreover, Φ is an anti-unitary operator from X^* onto X, that is Φ satisfies the conditions:

$$\Phi(\alpha f + \beta g) = \overline{\alpha}\Phi(f) + \overline{\beta}\Phi(g) \quad \textit{and} \quad (\Phi f, \Phi g)_X = (g, f)_{X^*}$$

for all $f, g \in X^*$ and $\alpha, \beta \in \mathbb{C}$.

Proof. The result follows immediately from the observation that $v_{f+g} = v_f + v_g$ and $v_{\alpha f} = \overline{\alpha} v_f$ for all $f \in X^*$ and $\alpha \in \mathbb{C}$. □

Corollary 8.3. *Every Hilbert space is reflexive.*

Proof. Let X be a Hilbert space as before. Let $\Phi : X^* \longrightarrow X$ and $\Phi^* : X^{**} \longrightarrow X^*$ be the anti-unitaries given by the Riesz Representation Theorem. If we put j and j^* be the inverse of Φ and Φ^* respectively, then j and j^* are clearly given by

$$(j(x))(y) := (y,x)_X \text{ and } (j^*(f))(g) = (g,f)_{X^*}$$

for $x, y \in X$ and $f, g \in X^*$.
Now if we let $i : X \longrightarrow X^{**}$ be the canonical embedding, that is, $(i(x))(f) := f(x)$ for $x \in X$ and $f \in X^*$, then it suffices to show that

$$i = j^* \circ j.$$

In fact, for $x \in X$ and $f \in X^*$, we have

$$(j^* \circ j(x))(f) = (f, j(x))_{X^*} = (\Phi(j(x)), \Phi(f))_X = (x, \Phi(f))_X = f(x) = (i(x))(f).$$

The proof is finished. □

Theorem 8.4. *Every bounded sequence in a Hilbert space has a weakly convergent subsequence.*

Proof. Let (x_n) be a bounded sequence in a Hilbert space X and M be the closed subspace of X spanned by $\{x_m : m = 1, 2...\}$. Then M is a separable Hilbert space.
Method I : Let $j_M : x \in M \mapsto j_M(x) := (\cdot, x) \in M^*$ be the mapping defined as in Corollary 8.3. Then $(j_M(x_n))$ is a bounded sequence in M^*. By Banach's result, Proposition 5.11, $(j_M(x_n))$ has a w^*-convergent subsequence $(j_M(x_{n_k}))$. Put $j_M(x_{n_k}) \xrightarrow{w^*} f \in M^*$, that is $j_M(x_{n_k})(z) \to f(z)$ for all $z \in M$. The Riesz Representation will assure that there is a unique element $m \in M$ such that $j_M(m) = f$. So we have $(z, x_{n_k}) \to (z, m)$ for all $z \in M$. In particular, if we consider the orthogonal decomposition $X = M \oplus M^{\perp}$, then $(x, x_{n_k}) \to (x, m)$ for all $x \in X$ and thus $(x_{n_k}, x) \to (m, x)$ for all $x \in X$. Then $x_{n_k} \to m$ weakly in X by using the Riesz Representation Theorem again.
Method II : We first note that since M is a separable Hilbert space, the second dual M^{**} is also separable by the reflexivity of M. So the dual space M^* is also separable (see Proposition1.5). Let $Q : M \longrightarrow M^{**}$ be the natural canonical mapping. To apply the Banach's result Proposition 5.11 for X^*, then $Q(x_n)$ has a w^*-convergent subsequence, says $Q(x_{n_k})$. This gives an element $m \in M$ such that $Q(m) = w^*\text{-}\lim_k Q(x_{n_k})$ because M is reflexive. So we have $f(x_{n_k}) = Q(x_{n_k})(f) \to Q(m)(f) = f(m)$ for all $f \in M^*$. Using the same argument as in **Method I** again, x_{n_k} weakly converges to m as desired. □

Remark 8.5. *It is well known that we have the following Theorem due to R. C. James (the proof is highly non-trivial):*

A normed space X is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence.

Theorem 8.4 can be obtained by the James's Theorem directly. However, Theorem 8.4 gives a simple proof in the Hilbert space case.

9. Operators on a Hilbert space

Throughout this section, all spaces are complex Hilbert spaces. Let $B(X, Y)$ denote the space of all bounded linear operators from X into Y. If $X = Y$, write $B(X)$ for $B(X, X)$.
Let $T \in B(X, Y)$. We will make use the following simple observation:

$$(Tx, y) = 0 \text{ for all } x \in X; y \in Y \quad \text{if and only if} \quad T = 0. \tag{9.1}$$

Therefore, the elements in $B(X, Y)$ are uniquely determined by the Eq.9.1, that is, $T = S$ in $B(X, Y)$ if and only if $(Tx, y) = (Sx, y)$ for all $x \in X$ and $y \in Y$.

Proposition 9.1. *Let $T \in B(X)$. Then we have*

(i): $T = 0$ *if and only if* $(Tx, x) = 0$ *for all* $x \in X$.
(ii): $\|T\| = \sup\{|(Tx, y)| : x, y \in X$ *with* $\|x\| = \|y\| = 1\}$.

Proof. It is clear that the necessary part in Part (i). Now we are going to the sufficient part in Part (i), that is we assume that $(Tx, x) = 0$ for all $x \in X$. This implies that we have

$$0 = (T(x + iy), x + iy) = (Tx, x) + i(Ty, x) - i(Tx, y) + (Tiy, iy) = i(Ty, x) - i(Tx, y).$$

So we have $(Ty, x) - (Tx, y) = 0$ for all $x, y \in X$. In particular, if we replace y by iy in the equation, then we get $i(Ty, x) - \bar{i}(Tx, y) = 0$ and hence we have $(Ty, x) + (Tx, y) = 0$. Therefore we have $(Tx, y) = 0$.
For part (ii) : Let $\alpha = \sup\{|(Tx, y)| : x, y \in X$ with $\|x\| = \|y\| = 1\}$. It is clear that we have $\|T\| \geq \alpha$. It needs to show $\|T\| \leq \alpha$.
In fact, for each $x \in X$ with $\|x\| = 1$, then by the Hahn-Banach Theorem, there is $f \in X^*$ with $\|f\| = 1$ such that $f(Tx) = \|Tx\|$. The Riesz Representation Theorem, we can find an element $y_f \in X$ with $\|y_f\| = \|f\| = 1$ so that we have $\|Tx\| = f(Tx) = (x, y_f) \leq \alpha$ for all $x \in X$ with $\|x\| = 1$. This implies that $\|T\| \leq \alpha$. The proof is finished. □

Proposition 9.2. *Let* $T \in B(X)$. *Then there is a unique element* T^* *in* $B(X)$ *such that*

$$(Tx, y) = (x, T^*y) \tag{9.2}$$

In this case, T^* *is called the adjoint operator of* T.

Proof. We first show the uniqueness. Suppose that there are S_1, S_2 in $B(X)$ which satisfy the Eq.9.2. Then $(x, S_1y) = (x, S_2y)$ for all $x, y \in X$. Eq.9.1 implies that $S_1 = S_2$.
Finally, we prove the existence. Note that if we fix an element $y \in X$, define the map $f_y(x) := (Tx, y)$ for all $x \in X$. Then $f_y \in X^*$. The Riesz Representation Theorem implies that there is a unique element $y^* \in X$ such that $(Tx, y) = (x, y^*)$ for all $x \in X$ and $\|f_y\| = \|y^*\|$. On the other hand, we have

$$|f_y(x)| = |(Tx, y)| \leq \|T\|\|x\|\|y\|$$

for all $x, y \in X$ and thus $\|f_y\| \leq \|T\|\|y\|$. If we put $T^*(y) := y^*$, then T^* satisfies the Eq.9.2. Also, we have $\|T^*y\| = \|y^*\| = \|f_y\| \leq \|T\|\|y\|$ for all $y \in X$. So $T^* \in B(X)$ with $\|T^*\| \leq \|T\|$ indeed. Hence T^* is as desired. □

Proposition 9.3. *Let* $T, S \in B(X)$. *Then we have*
(i): $T^* \in B(X)$ *and* $\|T^*\| = \|T\|$.
(ii): The map $T \in B(X) \mapsto T^* \in B(X)$ *is an isometric conjugate anti-isomorphism, that is,*

$$(\alpha T + \beta S)^* = \overline{\alpha}T^* + \overline{\beta}S^* \quad \textit{for all} \quad \alpha, \beta \in \mathbb{C}; \quad \textit{and} \quad (TS)^* = S^*T^*.$$

(iii): $\|T^*T\| = \|T\|^2$.

Proof. For Part (i), in the proof of Proposition 9.2, we have shown that $\|T^*\| \leq \|T\|$. And the reverse inequality clearly follows from $T^{**} = T$.
The Part (ii) follows from the adjoint operators are uniquely determined by the Eq.9.2 above.
For Part (iii), we always have $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. For the reverse inequality, let (x_n) be a sequence in B_X such that $\|Tx_n\| \to \|T\|$. Since

$$\|Tx_n\|^2 = (Tx_n, Tx_n) = (T^*Tx_n, x_n) \leq \|T^*Tx_n\|\|x_n\| \leq \|T^*T\|,$$

we have $\|T\|^2 \leq \|T^*T\|$. □

Example 9.4. *If* $X = \mathbb{C}^n$ *and* $D = (a_{ij})_{n \times n}$ *an* $n \times n$ *matrix, then* $D^* = (\overline{a_{ji}})_{n \times n}$. *In fact, notice that*

$$a_{ji} = (De_i, e_j) = (e_i, D^*e_j) = \overline{(D^*e_j, e_i)}.$$

So if we put $D^* = (d_{ij})_{n \times n}$, *then* $d_{ij} = (D^*e_j, e_i) = \overline{a_{ji}}$.

Example 9.5. *Let* $\ell^2(\mathbb{N}) := \{x : \mathbb{N} \to \mathbb{C} : \sum_{i=0}^{\infty} |x(i)|^2 < \infty\}$. *And put* $(x, y) := \sum_{i=0}^{\infty} x(i)\overline{y(i)}$.
Define the operator $D \in B(\ell^2(\mathbb{N}))$ *(called the unilateral shift) by*

$$Dx(i) = x(i-1)$$

for $i \in \mathbb{N}$ *and where we set* $x(-1) := 0$, *that is* $D(x(0), x(1), ...) = (0, x(1), x(2), ...)$.
Then D *is an isometry and the adjoint operator* D^* *is given by*

$$D^*x(i) := x(i+1)$$

for $i = 0, 1, ..,$ *that is* $D^*(x(0), x(1), ...) = (x(1), x(2), ...)$.
Indeed one can directly check that

$$(Dx, y) = \sum_{i=0}^{\infty} x(i-1)\overline{y(i)} = \sum_{j=0}^{\infty} x(j)\overline{y(j+1)} = (x, D^*y).$$

Note that D^* *is NOT an isometry.*

Example 9.6. *Let* $\ell^\infty(\mathbb{N}) = \{x : \mathbb{N} \to \mathbb{C} : \sup_{i\geq 0} |x(i)| < \infty\}$ *and* $\|x\|_\infty := \sup_{i\geq 0} |x(i)|$. *For each* $x \in \ell^\infty$, *define* $M_x \in B(\ell^2(\mathbb{N}))$ *by*

$$M_x(\xi) := x \cdot \xi$$

for $\xi \in \ell^2(\mathbb{N})$, *where* $(x \cdot \xi)(i) := x(i)\xi(i)$; $i \in \mathbb{N}$.
Then $\|M_x\| = \|x\|_\infty$ *and* $M_x^* = M_{\overline{x}}$, *where* $\overline{x}(i) := \overline{x(i)}$.

10. Bounded operators on a Hilbert space II

Throughout this section, all spaces are complex Hilbert spaces.

Definition 10.1. *Let* $T \in B(X)$ *and let* I *be the identity operator on* X. T *is said to be*
(i) : selfadjoint if $T^* = T$;
(ii) : normal if $T^*T = TT^*$;
(iii) : unitary if $T^*T = TT^* = I$.

Proposition 10.2. *We have*
(i) : Let $T : X \longrightarrow X$ *be a linear operator.* T *is selfadjoint if and only if*

$$(Tx, y) = (x, Ty) \quad \text{for all } x, y \in X. \tag{10.1}$$

(ii) : T *is normal if and only if* $\|Tx\| = \|T^*x\|$ *for all* $x \in X$.

Proof. The necessary part of Part (i) is clear.
Now suppose that the Eq.10.3 holds, it needs to show that T is bounded.
For using the Closed Graph Theorem, we have to show if the sequences $x_n \to x$ and $y_n \to y$, implies $Tx = y$. In fact, for any $z \in X$, we have $(z, Tx_n) \to (z, y)$ and the Eq.10.3 gives

$$(z, Tx_n) = (Tz, x_n) \to (Tz, x) = (z, Tx).$$

So we have $(z, Tx) = (z, y)$ for all $z \in X$ and thus $Tx = y$. Therefore the Closed Graph Theorem will imply T being bounded immediately.
For Part (ii), note that by Proposition 9.1, T is normal if and only if $(T^*Tx, x) = (TT^*x, x)$. So, Part (ii) follows from that

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = (TT^*x, x) = (T^*x, T^*x) = \|T^*x\|^2$$

for all $x \in X$. □

Proposition 10.3. *Let $T \in B(X)$. Then we have*

$$\ker T = (imT^*)^\perp \quad \text{and} \quad (\ker T)^\perp = \overline{imT^*}$$

where imT denotes the image of T.

Proof. The first equality is clearly follows from $x \in \ker T$ if and only if $0 = (Tx, z) = (x, T^*z)$ for all $z \in X$.
On the other hand, it is clear that we have $M^\perp = \overline{M}^\perp$ for any subspace M of X. This together with the first equality and Corollary7.5 will yield the second equality at once. □

Proposition 10.4. *Let $(E, \|\cdot\|)$ be a Banach space. Let M and N be the closed subspaces of E such that*

$$E = M \oplus N \qquad \ldots\ldots\ldots\ldots(*)$$

Define an operator $Q : E \longrightarrow E$ by $Q(y+z) = y$ for $y \in M$ and $z \in N$. Then Q is bounded. In this case, Q is called the projection with respect to the decomposition $(*)$.
Furthermore, if E is a Hilbert space, then $N = M^\perp$ (and hence $(*)$ is the orthogonal decomposition of E with respect to M) if and only if Q satisfies the conditions: $Q^2 = Q$ and $Q^* = Q$. And Q is called the orthogonal projection (or projection for simply) with respect to M.

Proof. For each $x \in E$, write $x = y + z$ for $y \in M$ and $z \in N$ with respect to the decomposition $(*)$ above. And put $\|x\|_1 = \|y\| + \|z\|$. Then $\|\cdot\|_1$ is a norm on E. It is clear that Q is bounded with respect to the norm $\|\cdot\|_1$. So, the result follows if two norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.
In fact, since $\|\cdot\| \leq \|\cdot\|_1$, then by the Open Mapping Theorem, it suffices to show that $\|\cdot\|_1$ is also a complete norm. Let (x_n) be a Cauchy sequence in E with respect to the norm $\|\cdot\|_1$. Write $x_n = y_n + z_n$ for $y_n \in M$ and $z_n \in N$. Then (y_n) and (z_n) both are the Cauchy sequences with respect to the original norm $\|\cdot\|$. Since M and N are closed, there are $y \in M$ and $z \in N$ such that $y_n \to y$ and $z_n \to z$ with respect to the norm $\|\cdot\|$. This gives $x_n = y_n + z_n \to y + z$ in the norm $\|\cdot\|_1$.
For the last assertion, we further assume that E is a Hilbert space.
It is clear from the definition of Q that $Q(y) = y$ and $Q(z) = 0$ for all $y \in M$ and $z \in N$. Thus we have $Q^2 = Q$.
Now if $N = M^\perp$, then for $y, y' \in M$ and $z, z' \in N$, we have

$$(Q(y+z), y'+z') = (y, y') = (y+z, Q(y'+z')).$$

So $Q^* = Q$.
The converse of the last statement follows from Proposition 10.3 at once because $\ker Q = N$ and $imQ = M$.
The proof is complete. □

Proposition 10.5. *When X is a Hilbert space, we put $\mathcal{M}$ the set of all closed subspaces of X and $\mathcal{P}$ the set of all orthogonal projections on X. Now for each $M \in \mathcal{M}$, let P_M be the corresponding projection with respect to the orthogonal decomposition $X = M \oplus M^\perp$. Then there is an one-one correspondence between $\mathcal{M}$ and $\mathcal{P}$ which is defined by*

$$M \in \mathcal{M} \mapsto P_M \in \mathcal{P}.$$

Furthermore, if $M, N \in \mathcal{M}$, then we have

(i) : $M \subseteq N$ if and only if $P_M P_N = P_N P_M = P_M$.
(ii) : $M \perp N$ if and only if $P_M P_N = P_N P_M = 0$.

Proof. It first follows from Proposition 10.4 that $P_M \in \mathcal{P}$.
Indeed the inverse of the correspondence is given by the following. If we let $Q \in \mathcal{P}$ and $M = Q(X)$, then M is closed because $M = \ker(I - Q)$ and $I - Q$ is bounded. Also it is clear that $X = Q(X) \oplus (I - Q)X$ with $\ker Q = M^\perp$. Hence M is the corresponding closed subspace of X, that is $M \in \mathcal{M}$ and $P_M = Q$ as desired.
For the final assertion, Part (i) and (ii) follow immediately from the orthogonal decompositions $X = M \oplus M^\perp = N \oplus N^\perp$ and together with the clear facts that $M \subseteq N$ if and only if $N^\perp \subseteq M^\perp$. ☐

11. Spectral Theory I

Definition 11.1. *Let E be a normed space and let $T \in B(E)$. The spectrum of T, write $\sigma(T)$, is defined by*

$$\sigma(T) := \{\lambda \in \mathbb{C} : \ T - \lambda I \ \textit{is not invertible in} \ \ B(E)\}.$$

Remark 11.2. *More precise, for a normed space E, an operator $T \in B(E)$ is said to be invertible in $B(E)$ if T is an linear isomorphism and the inverse T^{-1} is also bounded. However, if E is complete, the Open Mapping Theorem assures that the inverse T^{-1} is bounded automatically. So if E is a Banach space and $T \in B(E)$, then $\lambda \notin \sigma(T)$ if and only if $T - \lambda := T - \lambda I$ is an linear isomorphism. So λ lies in the spectrum $\sigma(T)$ if and only if $T - \lambda$ is either not one-one or not surjective.*
In particular, if there is a non-zero element $v \in X$ such that $Tv = \lambda v$, then $\lambda \in \sigma(T)$ and λ is called an eigenvalue of T with eigenvector v.
We also write $\sigma_p(T)$ for the set of all eigenvalue of T and call $\sigma_p(T)$ the point spectrum.

Example 11.3. *Let $E = \mathbb{C}^n$ and $T = (a_{ij})_{n\times n} \in M_n(\mathbb{C})$. Then $\lambda \in \sigma(T)$ if and only if λ is an eigenvalue of T and thus $\sigma(T) = \sigma_p(T)$.*

Example 11.4. *Let $E = (c_{00}(\mathbb{N}), \|\cdot\|_\infty)$ (note that $c_{00}(\mathbb{N})$ is not a Banach space). Define the map $T : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N})$ by*

$$Tx(k) := \frac{x(k)}{k+1}$$

for $x \in c_{00}(\mathbb{N})$ and $i \in \mathbb{N}$.
Then T is bounded, in fact, $\|Tx\|_\infty \le \|x\|_\infty$ for all $x \in c_{00}(\mathbb{N})$.
On the other hand, we note that if $\lambda \in \mathbb{C}$ and $x \in c_{00}(\mathbb{N})$, then

$$(T - \lambda)x(k) = (\frac{1}{k+1} - \lambda)x(k).$$

From this we see that $\sigma_p(T) = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. And if $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$, then $T - \lambda$ is an linear isomorphism and its inverse is given by

$$(T - \lambda)^{-1}x(k) = (\frac{1}{k+1} - \lambda)^{-1}x(k).$$

So, $(T - \lambda)^{-1}$ is unbounded if $\lambda = 0$ and thus $0 \in \sigma(T)$.
On the other hand, if $\lambda \neq 0$, then $(T - \lambda)^{-1}$ is bounded. In fact, if $\lambda = a + ib \neq 0$, for $a, b \in \mathbb{R}$, then $\eta := \min_k |\frac{1}{1+k} - a|^2 + |b|^2 > 0$ because $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. This gives

$$\|(T - \lambda)^{-1}\| = \sup_{k\in\mathbb{N}} |(\frac{1}{k+1} - \lambda)^{-1}| < \eta^{-1} < \infty.$$

It can now be concluded that $\sigma(T) = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$.

Proposition 11.5. *Let E be a Banach space and $T \in B(E)$. Then*

(i) : $I-T$ is invertible in $B(E)$ whenever $\|T\|<1$.
(ii) : If $|\lambda|>\|T\|$, then $\lambda\notin\sigma(T)$.
(iii) : $\sigma(T)$ is a compact subset of $\mathbb{C}$.
(iv) : If we let $GL(E)$ the set of all invertible elements in $B(E)$, then $GL(E)$ is an open subset of $B(E)$ with respect to the $\|\cdot\|$-topology.

Proof. Notice that since $B(E)$ is complete, Part (i) clearly follows from the following equality immediately:

$$(I-T)(I+T+T^2+\cdots\cdots+T^{N-1})=I-T^N$$

for all $N\in\mathbb{N}$.
For Part (ii), if $|\lambda|>\|T\|$, then by Part (i) , we see that $I-\frac{1}{\lambda}T$ is invertible and so is $\lambda I-T$. This implies $\lambda\notin\sigma(T)$.
For Part (iii), since $\sigma(T)$ is bounded by Part (ii), it needs to show that $\sigma(T)$ is closed.
Let $c\in\mathbb{C}\setminus\sigma(T)$. It needs to find $r>0$ such that $\mu\notin\sigma(T)$ as $|\mu-c|<r$. Note that since $T-c$ is invertible, then for $\mu\in\mathbb{C}$, we have $T-\mu=(T-c)-(\mu-c)=(T-c)(I-(\mu-c)(T-c)^{-1})$. Therefore, if $\|(\mu-c)(T-c)^{-1})\|<1$, then $T-\mu$ is invertible by Part (i). So if we take $0<r<\dfrac{1}{\|(T-c)^{-1}\|}$, then r is as desired, that is, $B(c,r)\subseteq\ \mathbb{C}\setminus\sigma(T)$. Hence $\sigma(T)$ is closed.
For the last assertion, let $T\in GL(E)$. Notice that for any $S\in B(E)$, we have $\|T-S\|\leq\|T\|\|I-T^{-1}S\|$. So if $\|S\|<\dfrac{1}{\|T^{-1}\|}$, then $T-S$ is invertible by Part (i). Therefore we have $B(T,\frac{1}{\|T^{-1}\|})\subseteq GL(E)$.
The proof is finished. □

Corollary 11.6. *If U is a unitary operator on a Hilbert space X, then $\sigma(U)\subseteq\{\lambda\in\mathbb{C}:|\lambda|=1\}$.*

Proof. Since $\|U\|=1$, we have $\sigma(U)\subseteq\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}$ by Proposition 11.5(ii).
Now if $|\lambda|<1$, then $\|\lambda U^*\|<1$. By using Proposition 11.5 again, we have $I-\lambda U^*$ is invertible. This implies that $U-\lambda=U(I-\lambda U^*)$ is also invertible and thus $\lambda\notin\sigma(U)$. □

Example 11.7. *Let $E=\ell^2(\mathbb{N})$ and $D\in B(E)$ be the right unilateral shift operator as in Example9.5. Recall that $Dx(k):=x(k-1)$ for $i\in\mathbb{N}$ and $x(-1):=0$. Then $\sigma_p(D)=\emptyset$ and $\sigma(D)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}$.*
We first claim that $\sigma_p(D)=\emptyset$.
Suppose that $\lambda\in\mathbb{C}$ and $x\in\ell^2(\mathbb{N})$ satisfy the equation $Dx=\lambda x$. Then by the definition of D, we have

$$x(k-1)=\lambda x(k)\qquad\cdots\cdots\cdots(*)$$

for all $k\in\mathbb{N}$.
If $\lambda\neq 0$, then we have $x(k)=\lambda^{-1}x_{k-1}$ for all $i\in\mathbb{N}$. Since $x(-1)=0$, this forces $x(k)=0$ for all i, that is $x=0$ in $\ell^2(\mathbb{N})$.
On the other hand if $\lambda=0$, the Eq.$(*)$ gives $x(k-1)=0$ for all k and so $x=0$ again.
Therefore $\sigma_p(D)=\emptyset$.
Finally, we are going to show $\sigma(D)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}$.
Note that since D is an isometry, $\|D\|=1$. Proposition 11.5 tells us that

$$\sigma(D)\subseteq\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}.$$

Notice that since $\sigma_p(D)$ is empty, it suffices to show that $D-\mu$ is not surjective for all $\mu\in\mathbb{C}$ with $|\mu|\leq 1$.
Now suppose that there is $\lambda\in\mathbb{C}$ with $|\lambda|\leq 1$ such that $D-\lambda$ is surjective.
We consider the case when $|\lambda|=1$ first.

Let $e_1 = (1, 0, 0, ...) \in \ell^2(\mathbb{N})$*. Then by the assumption, there is* $x \in \ell^2(\mathbb{N})$ *such that* $(D - \lambda)x = e_1$ *and thus* $Dx = \lambda x + e_1$*. This implies that*

$$x(k-1) = Dx(k) = \lambda x(k) + e_1(k)$$

for all $k \in \mathbb{N}$*. From this we have* $x(0) = -\lambda^{-1}$ *and* $x(k) = -\lambda^{-k}x(0)$ *for all* $k \geq 1$ *because since* $e_1(0) = 1$ *and* $e_1(k) = 0$ *for all* $k \geq 1$*. Also since* $|\lambda| = 1$*, it turns out that* $|x(0)| = |x(k)|$ *for all* $k \geq 1$*. As* $x \in \ell^2(\mathbb{N})$*, this forces* $x = 0$*. However, it is absurd because* $Dx = \lambda x + e_1$*.*
Now we consider the case when $|\lambda| < 1$*.*
Notice that by Proposition 10.3, we have

$$\overline{im(D-\lambda)}^{\perp} = \ker(D-\lambda)^* = \ker(D^* - \overline{\lambda}).$$

Thus if $D - \lambda$ *is surjective, we have* $\ker(D^* - \overline{\lambda}) = (0)$ *and hence* $\overline{\lambda} \notin \sigma_p(D^*)$*.*
Notice that the adjoint D^* *of* D *is given by the left shift operator, that is,*

$$D^*x(k) = x(k+1) \qquad \cdots\cdots\cdots (**)$$

for all $k \in \mathbb{N}$*.*
Now when $D^*x = \mu x$ *for some* $\mu \in \mathbb{C}$ *and* $x \in \ell^2(\mathbb{N})$*, by using Eq.(**), which is equivalent to saying that*

$$x(k+1) = \mu x(k)$$

for all $k \in \mathbb{N}$*. So as* $|\overline{\lambda}| = |\lambda| < 1$*, if we set* $x(0) = 1$ *and* $x(k+1) = \overline{\lambda}^k x(0)$ *for all* $k \geq 1$*, then* $x \in \ell^2(\mathbb{N})$ *and* $D^*x = \overline{\lambda}x$*. Hence* $\overline{\lambda} \in \sigma_p(D^*)$ *which leads to a contradiction.*
The proof is finished.

12. Spectral Theory II

Throughout this section, let H be a complex Hilbert space.

Lemma 12.1. *Let* $T \in B(H)$*. We have the following assertions.*

(i) : T *is selfadjoint if and only if* $(Tx, x) \in \mathbb{R}$ *for all* $x \in H$*.*

(ii) : If T *is selfadjoint, then* $\|T\| = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$*.*

Proof. Part (i) is clearly follows from Proposition9.1.
For Part (ii), if we let $a = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$, then it is clear that $a \leq \|T\|$. We now going to show the reverse inequality. For $x, y \in H$ with $\|x\| = \|y\| = 1$, since T is selfadjoint, one can directly check that

$$(T(x+y), x+y) - (T(x-y), x-y) = 4(Tx, y).$$

This implies that

$$|(Tx, y)| \leq \frac{a}{4}(\|x+y\|^2 + \|x-y\|^2) = \frac{a}{2}(\|x\|^2 + \|y\|^2) = a.$$

Since $\|T\| = \sup_{\|x\|=\|y\|=1} |(Tx, y)|$, we have $\|T\| \leq a$ as desired. The proof is finished. □

Lemma 12.2. *Let* $T \in B(H)$ *be a normal operator (recall that* $T^*T = TT^*$*). Then* T *is invertible in* $B(H)$ *if and only if there is* $c > 0$ *such that* $\|Tx\| \geq c\|x\|$ *for all* $x \in H$*.*

Proof. The necessary part is clear.
Now we are going to show the converse. We first to show the case when T is selfadjoint. It is clear that T is injective from the assumption. So by the Open Mapping Theorem, it remains to show that T is surjective.
In fact since $\ker T = \overline{imT^*}^{\perp}$ and $T = T^*$, we see that the image of T is dense in H.
Now if $y \in H$, then there is a sequence (x_n) in H such that $Tx_n \to y$. So (Tx_n) is a Cauchy sequence. From this and the assumption give us that (x_n) is also a Cauchy sequence. If x_n

converges to $x \in H$, then $y = Tx$. Therefore the assertion is true when T is selfadjoint.
Now if T is normal, then we have $\|T^*x\| = \|Tx\| \geq c\|x\|$ for all $x \in H$ by Proposition 10.2(ii). Therefore, we have $\|T^*Tx\| \geq c\|Tx\| \geq c^2\|x\|$. Hence T^*T still satisfies the assumption. Notice that T^*T is selfadjoint. So we can apply the previous case to know that T^*T is invertible. This implies that T is also invertible because $T^*T = TT^*$.
The proof is finished. □

Definition 12.3. *Let $T \in B(X)$. We say that T is positive, write $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$.*

Remark 12.4. *It is clear that a positive operator is selfadjoint by Lemma 12.1 at once. And all projections are positive.*

Proposition 12.5. *Let $T \in B(H)$. We have*

(i) : If $T \geq 0$, then $T + I$ is invertible.

(ii) : If T is self-adjoint, then $\sigma(T) \subseteq \mathbb{R}$. In particular, when $T \geq 0$, we have $\sigma(T) \subseteq [0, \infty)$.

Proof. For Part (i), we assume that $T \geq 0$. This implies that

$$\|(I+T)x\|^2 = \|x\|^2 + \|Tx\|^2 + 2(Tx, x) \geq \|x\|^2$$

for all $x \in H$. So the invertibility of $I + T$ follows from Lemma 12.2.
For Part (ii), we first claim that $T + i$ is invertible. Indeed, it follows from $(T+i)^*(T+i) = T^2 + I$ and Part (i) immediately.
Now if $\lambda = a + ib \in \sigma(T)$ where $a, b \in \mathbb{R}$ with $b \neq 0$, then $T - \lambda = -b(\frac{-1}{b}(T - a) + i)$ is invertible because $\frac{-1}{b}(T - a)$ is selfadjoint.
Finally we are going to show $\sigma(T) \subseteq [0, \infty)$ when $T \geq 0$. Notice that since $\sigma(T) \subseteq \mathbb{R}$, it suffices to show that $T - c$ is invertible if $c < 0$. Indeed, if $c < 0$, then we see that $T - c = -c(I + (\frac{-1}{c}T))$ is invertible by the previous assertion because $\frac{-1}{c}T \geq 0$.
The proof is finished. □

Remark 12.6. *In Proposition 12.5, we have shown that if T is selfadjoint, then $\sigma(T) \subseteq \mathbb{R}$. However, the converse does not hold. For example, consider $H = \mathbb{C}^2$ and*

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Theorem 12.7. *Let $T \in B(H)$ be a selfadjoint operator. Put*

$$M(T) := \sup_{\|x\|=1} (Tx, x) \quad \text{and} \quad m(T) = \inf_{\|x\|=1} (Tx, x).$$

For convenience, we also write $M = M(T)$ and $m = m(T)$ if there is no confusion. Then we have

(i) : $\|T\| = \max\{|m|, |M|\}$.

(ii) : $\{m, M\} \subseteq \sigma(T)$.

(iii) : $\sigma(T) \subseteq [m, M]$.

Proof. Notice that m and M are defined because (Tx, x) is real for all $x \in H$ by Lemma 12.1 (ii). Also Part(i) can be obtained by using Lemma 12.1 (ii) again.
For Part (ii), we first claim that $M \in \sigma(T)$ if $T \geq 0$. Notice that $0 \leq m \leq M = \|T\|$ in this case by Lemma 12.1. Then there is a sequence (x_n) in H with $\|x_n\| = 1$ for all n such that $(Tx_n, x_n) \to M = \|T\|$. Then we have

$$\|(T - M)x_n\|^2 = \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M(Tx_n, x_n) \leq \|T\|^2 + M^2 - 2M(Tx_n, x_n) \to 0.$$

So by Lemma 12.2 we have shown that $T - M$ is not invertible and hence $M \in \sigma(T)$ if $T \geq 0$.
Now for any selfadjoint operator T if we consider $T - m$, then $T - m \geq 0$. Thus we have $M - m = M(T - m) \in \sigma(T - m)$ by the previous case. It is clear that $\sigma(T - c) = \sigma(T) - c$ for all $c \in \mathbb{C}$. Therefore we have $M \in \sigma(T)$ for any self-adjoint operator.
We are now claiming that $m(T) \in \sigma(T)$. Notice that $M(-T) = -m(T)$. So we have $-m(T) \in \sigma(-T)$. It is clear that $\sigma(-T) = -\sigma(T)$. Then $m(T) \in \sigma(T)$.
Finally, we are going to show $\sigma(T) \subseteq [m, M]$.
Indeed, since $T - m \geq 0$, then by Proposition 12.5, we have $\sigma(T) - m = \sigma(T - m) \subseteq [0, \infty)$. This gives $\sigma(T) \subseteq [m, \infty)$.
On the other hand, similarly, we consider $M - T \geq 0$. Then we get $M - \sigma(T) = \sigma(M - T) \subseteq [0, \infty)$. This implies that $\sigma(T) \subseteq (-\infty, M]$. The proof is finished. □

13. Compact operators on a Hilbert space

Throughout this section, let H be a complex Hilbert space.

Definition 13.1. *A linear operator $T : H \to H$ is said to be compact if for every bounded sequence (x_n) in H, $(T(x_n))$ has a norm convergent subsequence.*
Write $K(H)$ for the set of all compact operators on H and $K(H)_{sa}$ for the set of all compact selfadjoint operators.

Remark 13.2. *Let U be the closed unit ball of H. It is clear that T is compact if and only if the norm closure $\overline{T(U)}$ is a compact subset of H. Thus if T is compact, then T is bounded automatically because every compact set is bounded.*
Also it is clear that if T has finite rank, that is $\dim\ imT < \infty$, then T must be compact because every closed and bounded subset of a finite dimensional normed space is equivalent to it is compact.

Example 13.3. *The identity operator $I : H \to H$ is compact if and only if $\dim H < \infty$.*

Example 13.4. *Let $H = \ell^2(\{1, 2...\})$. Define $Tx(k) := \frac{x(k)}{k}$ for $k = 1, 2....$ Then T is compact.*
In fact, if we let (x_n) be a bounded sequence in ℓ^2, then by the diagonal argument, we can find a subsequence $y_m := Tx_m$ of Tx_n such that $\lim_{m\to\infty} y_m(k) = y(k)$ exists for all $k = 1, 2...$ Let $L := \sup_n \|x_n\|_2^2$. Since $|y_m(k)|^2 \leq \frac{L}{k^2}$ for all m, k, we have $y \in \ell^2$. Now let $\varepsilon > 0$. Then one can find a positive integer N such that $\sum_{k\geq N} 4L/k^2 < \varepsilon$. So we have

$$\sum_{k\geq N} |y_m(k) - y(k)|^2 < \sum_{k\geq N} \frac{4L}{k^2} < \varepsilon$$

for all m. On the other hand, since $\lim_{m\to\infty} y_m(k) = y(k)$ for all k, we can choose a positive integer M such that

$$\sum_{k=1}^{N-1} |y_m(k) - y(k)|^2 < \varepsilon$$

for all $m \geq M$. Finally, we have $\|y_m - y\|_2^2 < 2\varepsilon$ for all $m \geq M$.

Theorem 13.5. *Let $T \in B(H)$. Then T is compact if and only if T maps every weakly convergent sequence in H to a norm convergent sequence.*

Proof. We first assume that $T \in K(H)$. Let (x_n) be a bounded sequence in H. Since H is reflexive, (x_n) is bounded by the Uniform Boundedness Theorem. So we can find a subsequence (x_j) of (x_n) such that (Tx_j) is norm convergent. Let $y := \lim_j Tx_j$. We claim that $y = \lim_n Tx_n$. Suppose not. Then by the compactness of T again, we can find a subsequence (x_i) of (x_n) such that Tx_i

converges to y' with $y \neq y'$. Thus there is $z \in H$ such that $(y, z) \neq (y', z)$. On the other hand, if we let x be the weakly limit of (x_n), then $(x_n, w) \to (x, w)$ for all $w \in H$. So we have

$$(y, z) = \lim_j (Tx_j, z) = \lim_j (x_j, T^*(z)) = (x, T^*z) = (Tx, z).$$

Similarly, we also have $(y, z') = (Tx, z)$ and hence $(y, z) = (y, z')$ that contradicts to the choice of z.
For the converse, let (x_n) be a bounded sequence. Then by Theorem 8.4, (x_n) has a weakly convergent subsequence. Thus $T(x_n)$ has a norm convergent subsequence by the assumption at once. So T is compact. □

Proposition 13.6. *Let $S, T \in K(H)$. Then we have*

(i) : $\alpha S + \beta T \in K(H)$ for all $\alpha, \beta \in \mathbb{C}$;
(ii) : TQ and $QT \in K(H)$ for all Q in $B(H)$;
(iii) : $T^* \in K(H)$.

Moreover $K(H)$ is normed closed in $B(H)$.
Hence $K(H)$ is a closed $*$-ideal of $B(H)$.

Proof. (i) and (ii) are clear.
For property (iii), let (x_n) be a bounded sequence. Then (T^*x_n) is also bounded. So TT^*x_n has a convergent subsequence $TT^*x_{n_k}$ by the compactness of T. Notice that we have

$$\|T^*x_{n_k} - T^*x_{n_l}\|^2 = (TT^*(x_{n_k} - x_{n_l}), x_{n_k} - x_{n_l})$$

for all n_k, n_l. This implies that $(T^*x_{n_k})$ is a Cauchy sequence and thus is convergent since (x_{n_k}) is bounded.
Finally we are going to show $K(H)$ is closed. Let (T_m) be a sequence in $K(H)$ such that $T_m \to T$ in norm. Let (x_n) be a bounded sequence in H. Then by the diagonal argument (see the proof in Proposition 5.11), there is a subsequence (x_{n_k}) of (x_n) such that $\lim_k T_m x_{n_k}$ exists for all m. Now let $\varepsilon > 0$. Since $\lim_m T_m = T$, there is a positive integer N such that $\|T - T_N\| < \varepsilon$. On the other hand, there is a positive integer K such that $\|T_N x_{n_k} - T_N x_{n_{k'}}\| < \varepsilon$ for all $k, k' \geq K$. So we can now have

$$\|Tx_{n_k} - Tx_{n_{k'}}\| \leq \|Tx_{n_k} - T_N x_{n_k}\| + \|T_N x_{n_k} - T_N x_{n_{k'}}\| + \|T_N x_{n_{k'}} - Tx_{n_{k'}}\| \leq (2L + 1)\varepsilon$$

for all $k, k' \geq K$ where $L := \sup_n \|x_n\|$. Thus $\lim_k Tx_{n_k}$ exists. It can now be concluded that $T \in K(H)$. The proof is finished. □

Corollary 13.7. *Let $T \in K(H)$. If $\dim H = \infty$, then $0 \in \sigma(T)$.*

Proof. Suppose that $0 \notin \sigma(T)$. Then T^{-1} exists in $B(H)$. Proposition 13.1 gives $I = TT^{-1} \in K(H)$. This implies $\dim H < \infty$. □

Proposition 13.8. *Let $T \in K(H)$ and let $c \in \mathbb{C}$ with $c \neq 0$. Then $T - c$ has a closed range.*

Proof. Notice that since $\frac{1}{c}T \in K(H)$, so if we consider $\frac{1}{c}T - I$, we may assume that $c = 1$. Let $S = T - I$. Let x_n be a sequence in H such that $Sx_n \to x \in H$ in norm. By considering the orthogonal decomposition $H = \ker S \oplus (\ker S)^\perp$, we write $x_n = y_n \oplus z_n$ for $y_n \in \ker S$ and $z_n \in (\ker S)^\perp$. We first claim that (z_n) is bounded. Suppose not. By considering a subsequence of (z_n), we may assume that we may assume that $\|z_n\| \to \infty$. Put $v_n := \dfrac{z_n}{\|z_n\|} \in (\ker S)^\perp$. Since $Sz_n = Sx_n \to x$, we have $Sv_n \to 0$. On the other hand, since T is compact, and (v_n) is bounded, by passing a subsequence of (v_n), we may also assume that $Tv_n \to w$. Since $S = T - I$, $v_n = Tv_n - Sv_n \to w - 0 = w \in (\ker S)^\perp$. Also from this we have $Sv_n \to Sw$. On the other hand,

we have $Sw = \lim_n Sv_n = \lim_n Tv_n - \lim_n v_n = w - w = 0$. So $w \in \ker S \cap (\ker S)^{\perp}$. It follows that $w = 0$. However, since $v_n \to w$ and $\|v_n\| = 1$ for all n. It leads to a contradiction. So (z_n) is bounded.
Finally we are going to show that $x \in imS$. Now since (z_n) is bounded, (Tz_n) has a convergent subsequence (Tz_{n_k}). Let $\lim_k Tz_{n_k} = z$. Then we have

$$z_{n_k} = Sz_{n_k} - Tz_{n_k} = Sx_{n_k} - Tz_{n_k} \to x - z.$$

It follows that $x = \lim_k Sx_{n_k} = \lim_k Sz_{n_k} = S(x - z) \in imS$. The proof is finished. □

Theorem 13.9. Fredholm Alternative Theorem : *Let $T \in K(H)_{sa}$ and let $0 \neq \lambda \in \mathbb{C}$. Then $T - \lambda$ is injective if and only if $T - \lambda$ is surjective.*

Proof. Since T is selfadjoint, $\sigma(T) \subseteq \mathbb{R}$. So if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $T - \lambda$ is invertible. So the result holds automatically.
Now consider the case $\lambda \in \mathbb{R} \setminus \{0\}$.
Then $T - \lambda$ is also selfadjoint. From this and Proposition 10.3, we have $\ker(T - \lambda) = (im(T - \lambda))^{\perp}$ and $(\ker(T - \lambda))^{\perp} = \overline{im(T - \lambda)}$.
So the proof is finished by using Proposition 13.8 immediately. □

Corollary 13.10. *Let $T \in K(H)_{sa}$. Then we have $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$. Consequently if the values $m(T)$ and $M(T)$ which are defined in Theorem 12.7 are non-zero, then both are the eigenvalues of T and $\|T\| = \max\limits_{\lambda \in \sigma_p(T)} |\lambda|$.*

Proof. It follows from the Fredholm Alternative Theorem at once. This together with Theorem 12.7 imply the last assertion. □

Example 13.11. *Let $T \in B(\ell^2)$ be defined as in Example 13.4. We have shown that $T \in K(\ell^2)$ and it is clear that T is selfadjoint. Then by Corollary 13.10 and Corollary 13.7, we see that $\sigma(T) = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$.*

Lemma 13.12. *Let $T \in K(H)_{sa}$ and let $E_\lambda := \{x \in H : Tx = \lambda x\}$ for $\lambda \in \sigma(T) \setminus \{0\}$, that is the eigenspace of T corresponding to λ. If we fix $\mu \in \sigma(T) \setminus \{0\}$ and put $I_\mu := \{\lambda \in \sigma(T) : |\lambda| = |\mu|\}$, then we have*

$$\dim \bigoplus_{\lambda \in I_\mu} E_\lambda < \infty.$$

Proof. We first notice that $\dim E_\lambda < \infty$ for all $\lambda \in \sigma_p(T) \setminus \{0\}$ because the restriction $T|E_\lambda$ is also a compact operator on E_λ.
On the other hand, since T is selfajoint, we also have $E_\lambda \perp E_{\lambda'}$ for $\lambda, \lambda' \in \sigma_p(T)$ with $\lambda \neq \lambda'$. Let $V := \bigoplus_{\lambda \in I_\mu} E_\lambda$. Suppose that $\dim V = \infty$. Then $|I_\mu| = \infty$. So, we can find an infinite sequence in I_μ such that $\lambda_m \neq \lambda_n$ for $m \neq n$. Now choose $v_n \in E_{\lambda_n}$ with $\|v_n\| = 1$ for each λ_n. Then $v_n \perp v_m$ for $n \neq m$. This implies that $\|Tv_n - Tv_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 = 2|\mu|^2 > 0$ for $m \neq n$. So (Tv_n) has no convergent subsequences which contradicts to T being compact. □

Theorem 13.13. *Let $T \in K(H)_{sa}$. And suppose that $\dim H = \infty$. Then $\sigma(T) = \{\lambda_1, \lambda_2, ...\} \cup \{0\}$, where (λ_n) is a sequence of real numbers with $\lambda_n \neq \lambda_m$ for $m \neq n$ and $|\lambda_n| \downarrow 0$.*

Proof. Note that since $\|T\| = \max(|M(T)|, |m(T)|)$ and $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$. So by Corollary 13.10, there is $|\lambda_1| = \max\limits_{\lambda \in \sigma_p(T)} |\lambda| = \|T\|$. Since $\dim E_{\lambda_1} < \infty$, then $E_{\lambda_1}^{\perp} \neq 0$. Then by considering the restriction of $T_2 := T|E_{\lambda_1}^{\perp} \neq 0$, there is $|\lambda_2| = \max_{\lambda \in \sigma_p(T_2)} |\lambda| = \|T_2\|$. Notice that $\lambda_2 \in \sigma_p(T)$ and $|\lambda_2| \leq |\lambda_1|$ because $\|T_2\| \leq \|T\|$. To repeat the same step, we can get a sequence (λ_n) such that $(|\lambda_n|)$ is decreasing.

Now we claim that $\lim_n |\lambda_n| = 0$.
Otherwise, there is $\eta > 0$ such that $|\lambda_n| \geq \eta$ for all n. If we let $v_n \in E_{\lambda_n}$ with $\|v_n\| = 1$ for all n. Notice that since $\dim H = \infty$ and $\dim E_\lambda < \infty$, for any $\lambda \in \sigma_p(T) \setminus \{0\}$, there are infinite many λ_n's. Then $w_n := \frac{1}{|\lambda_n|} v_n$ is a bounded sequence and $\|Tw_n - Tw_n\|^2 = \|v_n - v_m\|^2 = 2$ for $m \neq n$. This is a contradiction since T is compact. So $\lim_n |\lambda_n| = 0$.
Finally we need to check $\sigma(T) = \{\lambda_1, \lambda_2, ...\} \cup \{0\}$.
In fact, let $\mu \in \sigma_p(T)$. Since $|\lambda_n| \downarrow 0$, we can find a subsequence $n_1 < n_2 < ....$ of positive integers such that

$$|\lambda_1| = ... = |\lambda_{n_1}| > |\lambda_{n_1+1}| = ... = |\lambda_{n_2}| > |\lambda_{n_2+1}| = ... = |\lambda_{n_3}| > |\lambda_{n_3+1}| = ...$$

Then we can choose N such that $|\lambda_{n_N+1}| < |\mu| \leq |\lambda_{n_N}|$. Notice that by the construction of λ_n's implies $\mu = \lambda_j$ for some $n_{N-1} + 1 \leq j \leq n_N$.
The proof is finished. □

Theorem 13.14. *Let $T \in K(H)_{sa}$ and let (λ_n) be given as in Theorem 13.13. For each $\lambda \in \sigma_p(T) \setminus \{0\}$, put $d(\lambda) := \dim E_\lambda < \infty$. Let $\{e_{\lambda,i} : i = 1, ..., d(\lambda)\}$ be an orthonormal base for E_λ. Then we have the following orthogonal decomposition:*

$$H = \ker T \oplus \bigoplus_{n=1}^{\infty} E_{\lambda_n}. \tag{13.1}$$

Moreover $\mathcal{B} := \{e_{\lambda,i} : \lambda \in \sigma_p(T) \setminus \{0\}; i = 1, .., d(\lambda)\}$ forms an orthonormal base of $\overline{T(H)}$.
Also the series $\sum_{n=1}^{\infty} \lambda_n t_n$ norm converges to T, where $t_n(x) := \sum_{i=1}^{d(\lambda_n)} (x, e_{\lambda_n,i}) e_{\lambda_n,i}$, for $x \in H$.

Proof. Put $E = \bigoplus_{n=1}^{\infty} E_{\lambda_n}$. It is clear that $\ker T \subseteq E^\perp$. On the other hand, if the restriction $T_0 := T|E^\perp \neq 0$, then there exists an non-zero element $\mu \in \sigma_p(T_0) \subseteq \sigma_p(T)$ because $T_0 \in K(E^\perp)$. It is absurd because $\mu \neq \frac{1}{\lambda_i}$ for all i. So $T|E^\perp = 0$ and hence $E^\perp \subseteq \ker T$. So we have the decomposition (13.1). And from this we see that the family $\mathcal{B}$ forms an orthonormal base of $(\ker T)^\perp$. On the other, we have $(\ker T)^\perp = \overline{imT^*} = \overline{imT}$. Therefore, $\mathcal{B}$ is an orthonormal base for $\overline{T(H)}$ as desired.
For the last assertion, it needs to show that the series $\sum_{n=1}^{\infty} \lambda_n t_n$ converges to T in norm. Notice that if we put $S_m := \sum_{n=1}^{m} t_n$, then the decomposition (13.1), $\lim_{m\to\infty} S_m x = Tx$ for all $x \in H$. So it suffices to show that $(S_m)_{m=1}^{\infty}$ is a Cauchy sequence in $B(H)$. In fact we have

$$\|\lambda_{m+1} t_{m+1} + \cdots\cdots + \lambda_{m+p} t_{m+p}\| = |\lambda_{m+1}|$$

for all $m, p \in \mathbb{N}$ because $E_{\lambda_n} \perp E_{\lambda_m}$ for $m \neq n$ and $|\lambda_n|$ is decreasing. This gives that (S_n) is a Cauchy sequence since $|\lambda_n| \downarrow 0$. The proof is finished. □

Corollary 13.15. *$T \in K(H)$ if only if T can be approximated by finite rank operators.*

References

[1] J.B. Conway. A course in functional analysis, second edition, Springer-Verlag, (1990).
[2] W. Rudin. Functional analysis, second edition, McGraw-Hill Inc, (1991).

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